Projective bundles.

A projective bundle $Y \longrightarrow X$ is locally

$$U \times \mathbb{P}^r \longrightarrow U$$

similar to vector bundles which are locally
$U \times \mathbb{A}^r \longrightarrow U$.

Ex Trivial projective bundle

If $\mathcal{E} \cong \mathcal{O}_X^{r+1}$ then

$$X \times \mathbb{P}^r = \operatorname{Proj}(\mathcal{O}_X[x_0, \dots, x_r])$$

$$= \operatorname{Proj}(\operatorname{Sym} \mathcal{E}^\vee)$$

More generally if $\mathcal{E}$ is locally free, then

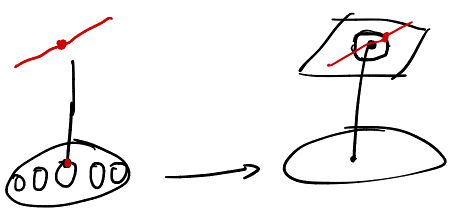
$$\mathbb{P}\mathcal{E} = \operatorname{Proj}(\operatorname{Sym} \mathcal{E}^\vee) \longrightarrow X \quad \text{is projectivisation of } \mathcal{E}.$$

Goal show every projective bundle is of form $\mathbb{P}\mathcal{E}$.

Tautological bundle

$$\begin{array}{ccc} \pi^*\mathcal{E} & & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathbb{P}\mathcal{E} & \xrightarrow{\pi} & X \end{array}$$

$\pi^*\mathcal{E} \to \mathbb{P}\mathcal{E}$ has tautological subbundle
its fiber at ξ is ξ.
Denote $\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1)$



Ex Projective bundles over $\mathbb{P}^1 = \mathbb{P}V$, V dim 2

v.b over $\mathbb{P}^1$ are of form $\bigoplus \mathcal{O}(a_i)$

Suppose $a_i \geq 0$. Let $\mathcal{E} = \bigoplus_{i=0}^{r} \mathcal{O}(-a_i)$

Set $W_i := H^0(\mathcal{O}_{\mathbb{P}^1}(a_i)) = \mathrm{Sym}^{a_i} V^* = k[s,t]_{\deg = a_i}$

$W = \bigoplus W_i = H^0(\mathcal{E}^\vee)$

$N = \dim W - 1 = \sum (a_i + 1) - 1 = r + \sum a_i$

As subbundles of $\mathbb{P}\mathcal{E}$, have rational curves

$C_i = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-a_i)) \cong \mathbb{P}^1$ degree a_i given by
(projectivization of line bundle) image of $(s,t) \mapsto (s^{a_i}; s^{a_i-1}t; \dots, t^{a_i})$

$\mathbb{P}^1 \to \mathbb{P}H^0(\mathcal{O}(a_i))$

$$\begin{tikzcd}
\underbrace{W = H^0(\mathcal{E}^*) \arrow[r, "\text{lift}"] & H^0(\pi^*\mathcal{E}^\vee) \arrow[r, "\text{dual of } \mathcal{O}(-1) \hookrightarrow \pi^*\mathcal{E}"] & H^0(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1))} \\
\bigoplus W_i \arrow[d, "\text{proj}"] \arrow[rr] & & H^0(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)) \arrow[d, "\text{restriction to } C_i"] \\
W_i = H^0(\mathcal{O}_{\mathbb{P}^1}(a_i)) \arrow[rr, "\cong"] & & H^0(\mathcal{O}_{\mathbb{P}\mathcal{O}_{\mathbb{P}^1}(-a_i)}(1))
\end{tikzcd}$$

So $W \to H^0(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1))$ is monomorphism

Have complete linear series $(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1), W)\big|_{\pi^{-1}(p)} = |\mathcal{O}_{\mathbb{P}^r}(1)|$
and induces a map

$\varphi : \mathbb{P}\mathcal{E} \to \mathbb{P}^N = \mathbb{P}W^*$

sending $\pi^{-1}(p) = \mathbb{P}^r$ to span of $\varphi_i(p)$

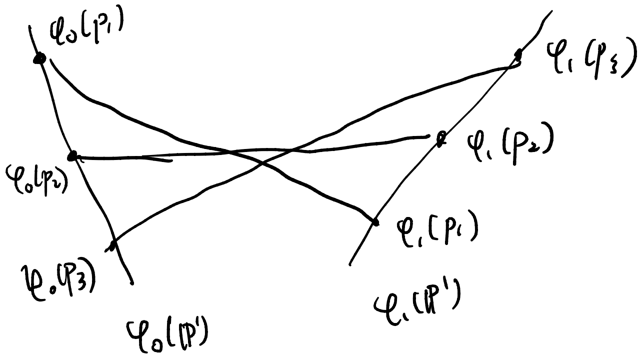
$$\varphi_i : \mathbb{P}^1 \to \mathbb{P} W_i^* = \mathbb{P} H^0(\mathcal{O}(a_i)) \subseteq \mathbb{P} W^*$$

$\deg a_i$ curve

$$p \mapsto \varphi_i(p)$$

<u>Fact</u> $(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1), W)$ is the linear system $|\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)|$.

<u>Ex</u> $d_0 = a_1 = 1$,



(rational normal scroll)

<u>Maps to projective bundles</u>

Suppose

$$\begin{tikzcd} & \mathcal{O} \arrow[r, squiggly] & \mathcal{E} \arrow[dl] \\ & \mathbb{P}\mathcal{E} \arrow[d, "\pi"] & \\ Y \arrow[ur, "\varphi"] \arrow[r, "p"] & X & \end{tikzcd}$$

commutative.

then φY gives a line subbundle of $p^*\mathcal{E}$.

Given by $\varphi^* \mathcal{O}_{\mathbb{P}\mathcal{E}}(-1)$

<u>Prop</u> Let $\pi : Y \to X$ smooth projective bundle.
then $Y = \mathbb{P}\mathcal{E}$ for some v.b on X

Pf Let
$$\begin{array}{c} \mathcal{U}\times\mathbb{P}^r \subseteq Y \\ \downarrow \\ \mathcal{U}, \end{array}$$

take divisor $\overline{\mathcal{U}\times H}$ then

get line bundle $\mathcal{L}$ s.t. $\mathcal{L}|_{Y_x} = \mathcal{O}_{\mathbb{P}^r}(1)$.

Let $\mathcal{E} = \pi_* \mathcal{L} \to X$ want map $\mathcal{L} \hookrightarrow \pi^* \mathcal{E}^*$

Have $\pi^*\pi_*\mathcal{L} \to \mathcal{L}$ natural map.

at each fiber get

$$\mathcal{E}_x \otimes_{\mathbb{P}\mathcal{E}_x} \mathcal{O}_{\mathbb{P}(\mathcal{E}_x)} \to \mathcal{O}_{\mathbb{P}(\mathcal{E}_x)}(1) \quad \text{over } \mathbb{P}(\mathcal{E}_x) \cong \mathbb{P}^r$$

so surjective, dual map injective.

induces map $Y \to \mathbb{P}\mathcal{E}$ fiberwise given by
$$\mathbb{P}^r \xrightarrow{|\mathcal{O}_{\mathbb{P}^r}(1)|} \mathbb{P}^r$$

check scheme isomorphism on local rings.

Chow ring of Proj bundle

If $Y = X \times \mathbb{P}^r$ then

$$\begin{aligned} A(Y) &= A(X) \otimes A(\mathbb{P}^r) \\ &= A(X)[\zeta]/(\zeta^{r+1}) \end{aligned}$$

Thm Let $\mathcal{E}$ v.b rank $r+1$ on X sm proj

Let $\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1))$ then

$$A(\mathbb{P}\mathcal{E}) \cong A(X)[\zeta]/(\zeta^{r+1} + c_1(\mathcal{E})\zeta^r + \dots + c_{r+1}(\mathcal{E}))$$

$$\cong \bigoplus_{i=1}^{r} A(X)\zeta^i \quad \text{as group.}$$

Pf

$\mathbb{P}\mathcal{E}$ here

$$\begin{array}{c} \mathbb{P}\mathcal{E} \\ \downarrow \pi \\ Y \end{array} \qquad 0 \to \underset{\text{rank } 1}{\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1)} \longrightarrow \underset{\text{rank } r+1}{p^*\mathcal{E}} \longrightarrow \underset{\text{rank } r}{Q} \longrightarrow 0$$

$$\pi^* c(\mathcal{E}) = c(\pi^*\mathcal{E}) = c(\mathcal{O}(1))\, c(Q)$$

$$= (1-\zeta)(1+c_1(Q)+\cdots+c_r(Q))$$

$$= 1+c_1(\mathcal{E})+\cdots+c_{r+1}(\mathcal{E})$$

$$\Longrightarrow c_1(\mathcal{E}) = c_1(Q) - \zeta$$

$$c_2(\mathcal{E}) = c_2(Q) - c_1(Q)\zeta$$

$$\vdots$$

$$c_{r+1}(\mathcal{E}) = -c_r(Q)\zeta$$

$\Longrightarrow$ get relation $\zeta^{r+1} + \cdots + c_{r+1}(\mathcal{E}) = 0.$

Fact $A(\mathbb{P}\mathcal{E}) \cong \bigoplus A(X)\zeta^i$

Corollary $\mathbb{P}\mathcal{E} \to X$, $\mathbb{P}\mathcal{E}' \to X$ isomorphic iff $\mathcal{E}', \mathcal{E}$ differ by line bundle

$$\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1) \cong \pi^*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}\mathcal{E}'}(-1)$$

Ex $\mathbb{P}^r$ parametrizes lines in $\mathbb{A}^{r+1}$

what if we replace with $Gr(k,r)$?

special case:

Let $\Phi \subseteq G \times \mathbb{P}^n$ universal k-plane,

then

$$A(\Phi) = A(G)[\zeta]/(\zeta^{k+1} - \sigma_1 \zeta^k + \sigma_{1,1}\zeta^{k-1} + \cdots + (-1)\sigma_{1,\dots,1})$$

where ζ is the tautological class.

<u>special case</u> universal hyperplane:

$$\Phi = \{(H,p) \in \mathbb{P}^{n*} \times \mathbb{P} : p \in H\}.$$

$$A(\Phi) = \mathbb{Z}[h,\zeta]/(h^{n+1}, \zeta^n - h\zeta^{n-1} + \cdots + (-1)^n h^n).$$